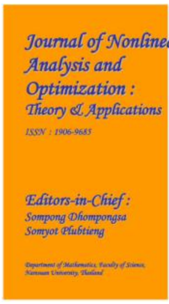
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 14, Issue. 2, No. 1: 2023
ISSN : **1906-9685**



DUAL SPACE OF A CONE NORMED SPACES

O. Lavanya, M.Sc., Assistant Professor, PG & Research Department of Mathematics Marudhar Kesari Jain College for Women, Vaniyambadi, Thirupattur District, Tamilnadu, India
Email: lavanyalavan218@gmail.com

Abstract:

Cone normed spaces are the generalization of the normed spaces with many authors adjusting the theory to the classical one. In this chapter, we explore and study some properties of the space of all continuous linear mappings between cone normed spaces and this allows us to define the concept of dual in the setting of cone normed spaces.

Keywords: Continuous linear mappings, Cone norm, Dual space

1. Introduction and Preliminaries:

Functional Analysis is a branch of mathematical analysis that studies functionals (a mapping acting between a vector space and a field $\mathbb{F}$= ($\mathbb{R}$/$\mathbb{C}$)) such as metric, norm, inner product and the spaces on which they act such as metric spaces, normed spaces, and inner product spaces in order to study their topological properties. It is well known that metric and norm structures play a pivotal role in functional analysis. In the early nineteenth century, Frechet became the first person to introduce the distance function called metric on a non-empty set X. Later Banach, Hahn and Weiner independently define a norm on a vector space X. Rzepecki introduced a notion of cone metric and a cone norm where the author replaced the real numbers as the co-domain of both the metric and norm by the real Banach space ordered by a cone. The concept serves as the generalization of both the metric and norm since the real numbers arealso complete with respect to the norm defined on them.many authors have used this concept in exploring more of its properties and applicability by attempting to adjust the theory of cone norm to the classical one. The aim of this paper is to study about the dual space of a cone normed space.

Definition 1-Pointed Cone

A nonempty subset P of a Banach space F is said to be a cone if

1. $P + P \subseteq P$ Closure under vector addition
2. $\mu P \subseteq P \; \forall \; \mu \geq 0$ Closure under scalar multiplication

In addition, $P \cap \{-P\} = \{0\}$, then P is called a Pointed Cone.

Remark:

For a cone, $P \subseteq F$, where F is a Banach space, we define a partial ordering $\leq$ with respect to P by $a \leq b$ if and only if $b - a \in P$ while $a \ll b$ if and only if $b - a \in int\ P$, where $int\ P$ denote the interior of P.

Observe that $b - a \in int\ P$ implies $b - a \in P$ but the reverse is not always the case. Thus, $a \ll b \Longrightarrow a \leq b$.

Definition 2.

Let the real vector space be X. Suppose that the mapping $\|\cdot\|_c : X \longrightarrow F$ where F is a real Banach space, satisfies:

a) $\|a\|_c \geq 0 \; \forall a \in X$ (1)

b) $\|a\|_c = 0 \Leftrightarrow a = 0 \; \forall a \in X$ (2)

c) $\|\alpha a\|_c = |\alpha|\|a\|_c \ \forall a \in X$ and $\alpha \in \mathbb{R}$ (3)

d) $\|a+b\|_c \le \|a\|_c + \|b\|_c \ \forall a, b \in X.$ (4)

where $\|\cdot\|_c$ is said to be cone norm on X and $(X, \|\cdot\|_c)$ is a cone normed space.

Definition 3.

Let $(X, \|\cdot\|_c)$ be a cone normed space. A sequence $(x_n) \in X$ is said to be

1. Cauchy, if for every $0 \ll c$ with $c \in F$ there exists $N \in \mathbb{N}$ such that
   a. $\left\|x_n - x_{n_0}\right\|_c \ll c \ \forall\ n, n_0 \ge \mathbb{N}$
2. Convergent to a, if for every $0 \ll c$ with $c \in F$ there exists $N \in \mathbb{N}$ such that
   $$\|x_n - a\|_c \ll c \ \forall n \ge \mathbb{N}$$

$(X, \|\cdot\|_c)$ is said to be complete cone normed space or cone Banach space if every Cauchy sequence converge.

2. SPACES OF CONTINOUS LINEAR MAPPINGS ON CONE NORMED SPACE

Consider the two cone normed spaces be $X_c = (X, \|\cdot\|_c)$,$Y_c = (Y, \|\cdot\|_c)$ and $(X, \|\cdot\|)$ be the classical normed space. Then $\mathcal{L}(X_c, Y_c)$ be the linear space of all continuous linear mappings from X_c to Y_c, i.e.$f: X_c \longrightarrow Y_c$ such that $f \in \mathcal{L}(X_c, Y_c)$.

Definition 4-Cone bounded map

Let X_c and Y_c be two cone normed spaces and $f: X_c \longrightarrow Y_c$ be a linear mapping. Then, f is said to be cone bounded if there exists $0 \ll M$ such that $\|f(a)\|_c \le M\|a\|_c$ for all $a \in X_c$.

Proposition 1.

Let $\left(X, \|\cdot\|_p\right)$ be a cone normed space, and $a \in X$, $0 \ll c$. Then

$$b \in \overline{B_c(a)} \Leftrightarrow (\exists\ \{z_n\} \subseteq B_c(a);\ z_n \longrightarrow b)$$

Proof:

Let $b \in \overline{B_c(a)}$.

Then for any positive integer n,

$$z_n \in B_{\frac{c}{2^n}}(b) \cap B_c(a) \neq \phi$$

We obtain, $z_n \longrightarrow b$ as $n \longrightarrow \infty$.

Suppose that $\{z_n\} \in B_c(a)$ is a sequence that $z_n \longrightarrow b$ as $n \longrightarrow \infty$.

Let W be an open set and that W consists of b.

There is $0 \ll p$ such that $B_p(b) \subseteq W$.

Choose the positive integer n, such that

$$\|z_n - b\|_p \ll p$$

Hence, $z_n \in B_p(b)$ and

$$W \cap B_c(a) \neq \phi$$

So that, $b \in \overline{B_c(a)}$

Hence, the proposition.

Lemma 1.

Let the two cone normed spaces be X_c, Y_c and $f: X_c \longrightarrow Y_c$ is a linear mapping. (Continuity at a point) For some fixed $a_0 \in X$and given $0 \ll c$ there is a $0 \ll t$ such that $\|f(a) - f(a_0)\|_c \ll c$ whenever $\|a - a_0\|_c \ll t$. Then (Continuity at zero) For $0 \ll c$ there is a $0 \ll t$ such that $\|f(a)\|_c \ll c$ whenever $a \in X$ and $\|a\|_c \ll t$.

Proof:

First, we assume that f has the property of continuity at a point.

For some $a_0 \in X$ and any $0 \ll c$ we can choose $0 \ll t$ such that $\|f(a) - f(a_0)\|_c \ll c$ whenever $\|a - a_0\|_c \ll t$.

Then for any $u \in X$ with $\|u\|_c \ll t$ we have $\|f(u + a_0) - f(a_0)\|_c \ll c$ because $\|(u + a_0) - a_0\|_c \ll t$.

But f is linear and hence $\|fu\|_c \ll c$ whenever $\|u\|_c \ll t$.

Hence, the Lemma

Lemma 2.

Let the two cone normed spaces be X_c and Y_c and $f: X_c \longrightarrow Y_c$ is a linear mapping. Then $f \in \mathcal{L}(X_c, Y_c)$, if and only if f is cone bounded.

Proof:

Assume that f is cone bounded, then there exists $0 \ll M$ such that $\|f(a)\|_c \leq M\|a\|_c$ for all $a \in X_c$.

Let $0 \ll c$ be given and consider an arbitrary $a_0 \in X$, then we can choose $t = \frac{c}{M} \gg 0$ such that $\|a - a_0\|_c \ll t$ for all $a \in X$. Now, for any $a \in X$

$$\|f(a) - f(a_0)\|_c = \|f(a - a_0)\|_c, \text{ since } f \text{ is linear}$$
$$\leq M\|(a - a_0)\|_c, \text{ since } f \text{ is cone bounded}$$
$$\ll M \times t = c$$

But a_0 was arbitrary, hence f is continuous.

Conversely,

Suppose that $f \in \mathcal{L}(X_c, Y_c)$ and we know that the continuity at a point implies continuity at zero. Thus, given $0 \ll c$ there is a $0 \ll t$ such that $\|f(a)\|_c \leq c$ whenever $a \in X$ and $\|a\|_c \ll t$.

Take any point $z \in X$ with $z \neq 0$ and set $a = \frac{t}{\|z\|_c} z$, then $\|a\|_c = t \Longrightarrow \|a\|_c \ll t$ and

$$\|f(a)\|_c = \left\| f\left(\frac{t}{\|z\|_c} z\right) \right\|_c = \frac{t}{\|z\|_c} \|f(z)\|_c \ll c$$

Thus, $\quad \|f(z)\|_c \ll \frac{c}{t} \|z\|_c \leq M\|z\|_c \; \forall \, z \in X.$

Definition 5. Non-Linear Scalarization function

Let Y be a topological vector space, P be a cone and a fixed $e \in int\, P$. A function $\xi_e: Y \longrightarrow \mathbb{R}$ defined by,

$$\xi_e(b) = inf\{r \in \mathbb{R}: b \in re - P\}, \forall \, b \in Y.$$

is called non-linear scalarization function.

Lemma 3.

Let$(X, \|\cdot\|_c)$ be a cone normed space. Then $\|\cdot\|: X \longrightarrow [0, \infty)$ defined by $\|\cdot\|: \xi_e \circ \|\cdot\|_c$ is a norm.

Proof:

We know that,

$$\|a\|_c = 0 \Leftrightarrow a = 0 \; \forall a \in X$$

Since $\xi_e(\cdot)$ is positively homogeneous.

And, by equation (3)

$$\|\alpha a\|_c = \xi_e(\|\alpha a\|_c) = \xi_e(|\alpha|\|a\|_c) = |\alpha|\xi_e(\|a\|_c) = |\alpha|\|a\|_c$$

Applying (4), we have,

$$\xi_e(\|a + b\|_c) \leq \xi_e(\|a\|_c + \|b\|_c) \leq \xi_e(\|a\|_c) + \xi_e(\|b\|_c)$$

for all $a, b \in X$

That is, $\|\cdot\|_c$ satisfies the triangle inequality.

Hence, $\|\cdot\|_c$ is a norm.

Proposition 2.

Let the two cone normed spaces be X_c and Y_c, and f is a linear mapping from X_c to Y_c. If f is continuous from X_c to Y_c, then it is continuous from normed spaces $(X, \|\cdot\|)$ to $(Y, \|\cdot\|)$.

Proof:

Assume that f is continuous from X_c to Y_c, by lemma (2), it is cone bounded.

Thus,

$$\|f(a)\|_c \leq M\|a\|_c$$

for all $a \in X_c$ and $0 \ll M \in F$.

Now,

$$\|f(a)\| = \xi_e \circ \|f(a)\|_c, \qquad \text{by lemma (3)}$$

$\|f(a)\| \leq \xi_e \circ (M\|a\|_c)$
$\|f(a)\| = M(\xi_e \circ \|a\|_c)$
$\|f(a)\| = M\|a\|$

Since continuity and boundedness of a linear map are equivalent in a classical normed space, and f is bounded, hence f is continuous.

Theorem 1.

Let the two cone normed spaces be X_c and Y_c. For each $f \in \mathcal{L}(X_c, Y_c)$, set

$$\|f\|_c = sup\{\|f(a)\|_c : \|a\|_c \ll 1\}$$

Then the following conditions holds

1. $\|f\|_c$ is a cone norm on $\mathcal{L}(X_c, Y_c)$.
2. If $(Y, \|\cdot\|_c)$ is a cone Banach space. Then $\mathcal{L}(X_c, Y_c)$ is a cone Banach space.

Proof:

1. Clearly $\|0\|_c = 0$. Now if $\|f\|_c = 0$

$\|f\|_c = sup\{\|f(a)\|_c : \|a\|_c \ll 1\} = 0.$
$\Longrightarrow \quad \|f(a)\|_c = 0$
$\Longrightarrow \quad f(a) = 0$
$\Longrightarrow \quad f = 0$

For the translation invariance, let $\alpha \in \mathbb{R}$ then

$$\|\alpha f\|_c = sup\{\|\alpha f(a)\|_c : \|a\|_c \ll 1\}$$
$$\|\alpha f\|_c = sup\{|\alpha|\|f(a)\|_c : \|a\|_c \ll 1\}$$
$$= |\alpha| sup\{\|f(a)\|_c : \|a\|_c \ll 1\}$$
$$= |\alpha|\|f\|_c$$

For the triangle inequality, we have for any $f, g \in \mathcal{L}(X_c, Y_c)$ that

$$\|f + g\|_c = sup\{\|f(a) + g(a)\|_c : \|a\|_c \ll 1\}$$
$$\leq sup\{\|f(a)\|_c + \|g(a)\|_c : \|a\|_c \ll 1\}$$
$$= sup\{\|f(a)\|_c : \|a\|_c \ll 1\} + sup\{\|g(a)\|_c : \|a\|_c \ll 1\}$$
$$= \|f\|_c + \|g\|_c$$

Hence $\|f\|_c$ is a cone norm on $\mathcal{L}(X_c, Y_c)$.

2. From (i), $\|f\|_c$ is a cone norm on $\mathcal{L}(X_c, Y_c)$.

To show that the space $\mathcal{L}(X_c, Y_c)$ is complete if $(Y, \|\cdot\|_c)$ is complete.

Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}(X_c, Y_c)$.

Then, given $0 \ll c$ with $c \in F$, there exist $N \in \mathbb{N}$ such that,

$$\left\|f_n - f_{n_0}\right\|_c \ll c \; \forall \, n, n_0 \geq N$$

Let $a \in X$, then,

$$\left\|f_n(a) - f_{n_0}(a)\right\|_c = \left\|\left(f_n - f_{n_0}\right)(a)\right\|_c \leq \left\|f_n - f_{n_0}\right\|_c \|a\|_c \ll c\|a\|_c \; \forall \, a \in X \tag{5}$$

Hence, $\{f_n(a)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists $b \in Y$ such that,

$$f_n(a) \longrightarrow b \text{ as } n \longrightarrow \infty$$

Setting $f(a) \longrightarrow b$. We shows that $f \in \mathcal{L}(X_c, Y_c)$ and $f_n \longrightarrow f$. Let $a_1, a_2 \in X$ and $\alpha, \beta \in \mathbb{R}$. Then,

$$f(\alpha a_1 + \beta a_2) = \lim_{n \to \infty} f_n\,(\alpha a_1 + \beta a_2)$$
$$f(\alpha a_1 + \beta a_2) = \lim_{n \to \infty} \left(\alpha f_n(a_1) + \beta f_n(a_2)\right)$$
$$= \alpha \lim_{n \to \infty} f_n(a_1) + \beta \lim_{n \to \infty} f_n(a_2)$$
$$= \alpha f(a_1) + \beta f(a_2)$$

Thus, f is linear. Now taking the limit as $n \longrightarrow \infty$ of equation (5) above, we get

$$\|(f_n - f)(a)\|_c = \|f_n(a) - f(a)\|_c \ll c\|a\|_c \; \forall n \geq N \; \& \; a \in X$$

Thus, $f_n - f$ is a cone bounded mapping for all $n \geq N$. Hence $f_n - f \in \mathcal{L}(X_c, Y_c)$ by lemma 2, which implies that $f = f_n - (f_n - f) \in \mathcal{L}(X_c, Y_c)$.

Hence,

$$\|f_n - f\|_c = sup\{\|f_n(a) - f(a)\|_c : \|a\|_c \ll 1\} \ll sup\{\|a\|_c c : \|a\|_c \ll 1\} \ll c \; \forall n \geq N$$

which implies that, $f_n \longrightarrow f$ as $n \longrightarrow \infty$.

3. DUAL SPACE OF A CONE NORMED SPACE

If $Y_c = \mathbb{R}_c$, where $\mathbb{R}_c = (\mathbb{R}, \|\cdot\|_c)$, then $\mathcal{L}(X_c, Y_c) = \mathcal{L}(X_c, \mathbb{R}_c)$. We observe that $\|\cdot\|_c = |\cdot|$ if $\|\cdot\|_c$ is defined on the space of real numbers $\mathbb{R}$. Hence, we define the dual space X_c^* of a cone normed space X_c such that

$$X_c^* = \{f: (X, \|\cdot\|_c) \longrightarrow (\mathbb{R}, |\cdot|),\ where\ f\ is\ linear\ and\ continuous\}$$

Since $\mathbb{R}$ is complete with respect to $\|\cdot\|_c = |\cdot|$, X_c^* is a cone Banach space. Thus, X_c^* is a dual of a cone normed space $(X, \|\cdot\|_c)$.

Definition 6.

Let X be a linear space. A mapping $\rho: X \longrightarrow \mathbb{R}$ is said to be sublinear functional if the following conditions are satisfied:

a. $\rho(a + b) \leq \rho(a) + \rho(b)\ \forall a, b \in X$ (Triangle inequality)

b. $\rho(\mu a) = \mu\rho(a), \mu \geq 0$ (Positive homogeneous)

Definition 7-Semi-cone norm

A semi-cone norm ρ_c on a linear space. X is a mapping $\rho_c: X \longrightarrow F$ such that,

1. $\rho_c(a) \geq 0$ and $\rho_c(0) = 0$
2. $\rho_c(\mu a) = |\mu|\rho_c(a)$
3. $\rho_c(a + b) \leq \rho_c(a) + \rho_c(b)$

A semi-cone norm ρ_c is a sublinear functional only if $F = \mathbb{R}$.

Example 1.

1. Every cone norm is a semi-cone norm but the converse is not the case.

2. Let $F = \mathbb{R}^2$ and P be a positive cone define by $P = \{a_1, a_2 | a_1 \geq 0, a_2 \geq 0\}$ with a coordinatewise ordering and $X = \mathbb{R}^2$. A mapping $\rho_c: X \longrightarrow F$ defined by $\rho_c(a_1, a_2) = \left(\frac{|a_1 - a_2|}{2}, |a_1 - a_2|\right)$ is a semi-cone norm on X which is nor a cone norm.

Proof:

1. This is obvious.
2. Let $(a_1 - a_2) = (0,0)$, then $\rho_c(0,0) = (0,0)$. For any $\alpha \in \mathbb{R}$ we have

$$\rho_c\big(\alpha(a_1, a_2)\big) = \rho_c(\alpha a_1, \alpha a_2)$$

$$= \left(\frac{|\alpha a_1 - \alpha a_2|}{2}, |\alpha a_1 - \alpha a_2|\right)$$

$$= \frac{|\alpha||a_1 - a_2|}{2}, |\alpha||a_1 - a_2|$$

$$= |\alpha|\rho_c(a_1, a_2)$$

For the triangle inequality,

Let $(a_1, a_2), (b_1, b_2) \in X$, then

$$\rho_c(a_1, a_2) + (b_1, b_2) = \rho_c\big((a_1 + b_1), (a_2 + b_2)\big)$$

$$= \left(\frac{|(a_1+b_1)-(a_2+b_2)|}{2}, |(a_1 + b_1) - (a_2 + b_2)|\right)$$

$$= \left(\frac{|(a_1-a_2)+(b_1-b_2)|}{2}, |(a_1 - a_2) + (b_1 - b_2)|\right)$$

$$= \left(\frac{|(a_1-a_2)|}{2}, |(a_1 - a_2)|\right) + \left(\frac{|(b_1-b_2)|}{2}, |(b_1 - b_2)|\right)$$

$$\rho_c(a_1, a_2) + (b_1, b_2) = \rho_c(a_1, a_2) + \rho_c(b_1, b_2)$$

Hence, ρ_c is a semi-cone norm. We see that ρ_c is not a cone norm.

If $\rho_c(a_1, a_2) = (0,0)$ then

$$\left(\frac{|(a_1-a_2)|}{2}, |(a_1 - a_2)|\right) = (0,0)$$

$$\frac{|(a_1-a_2)|}{2} = 0\ \&|(a_1 - a_2)| = 0$$

which implies that

$$|a_1 - a_2| = 0 \Longrightarrow a_1 - a_2 = 0 \Longrightarrow a_1 = a_2.$$

Lemma 4.

Let$f \in X_c^*$ and $a \in X_c$ be defined such that $\rho(a) = \|f\|_c\|a\|_c$. Then ρ is a sublinear functional.

Proof:

First to prove that ρ is a well defined functional. Since,

$$\rho(a) = \|f\|_c\|a\|_c = \sup_{a\in X}\frac{|f(a)|}{\|a\|_c}\|a\|_c = \sup_{a\in X}|f(a)|$$

Hence, ρ is a real valued functional.

For triangle inequality, let $a, b \in X$ then

$$\begin{aligned}\rho(a+b) &= \|f\|_c\|a+b\|_c \\ &\le \|f\|_c(\|a\|_c + \|b\|_c) \\ &= \|f\|_c\|a\|_c + \|f\|_c\|b\|_c \\ &= \rho(a) + \rho(b)\end{aligned}$$

For any $\mu \geq 0$, we have

$$\begin{aligned}\rho(\mu a) &= \|f\|_c\|\mu a\|_c \\ &= |\mu|\|f\|_c\|a\|_c \\ &= |\mu|\rho(a) \\ &= \mu\rho(a)\end{aligned}$$

Since $\mu \geq 0$.

Conclusion:

This paper have clearly shown that the concept of duality in cone normed space is achievable and proved some of its properties.

References:

1. Eshaghi GM, Ramezani M, Khodaei H, Baghani H. Cone normed spaces. Caspian Journalof Mathematical Sciences (CJMS). 2012;1(1).
2. Fréchet MM. Sur quelques points du calcul fonctionnel, Rendiconti del Circolo Matematico di Palermo (1884-1940). 1906;22(1):1–72.
3. İlkhan M, Alp PZ, Kara EE. On the spaces of linear operators acting between asymmetric cone normed spaces, Mediterranean Journal of Mathematics. 2018;15(3):136. Available: https://doi.org/10.1007/s00009-018-1182-0
4. Kreyszig E. Introductory functional analysis with applications, Wiley New York; 1978.
5. Rzepecki B. On fixed point theorems of Maia type, Publications de l'Institut Mathématique. 1980;28(42): 179–186.
6. Samanta T, Roy S, Dinda B. Cone normed linear spaces. arXiv preprint arXiv:1009.2172; 2010.
7. Sarkar K, Tiwary K. Fixed point theorem in cone banach spaces. International Journal of Statistics and Applied Mathematics. 2018;3(4):143–146.
8. Tamang P, Bag T. Some fixed point results in fuzzy cone normed linear space. Journal of the Egyptian Mathematical Society. 2019;27(1):46. Available: https://doi.org/10.1186/s42787-019-0046-6
9. Wiener N. Limit in terms of continuous transformation. Bulletin de la Société Mathématique de France. 1922;50:119–134.